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Stratifications of cellular patterns: hysteresis and convergence

C. Oguey[1,a], N. Rivier[2], and T. Aste[3]

[1] LPTM[b], Université de Cergy-Pontoise, 95031 Cergy-Pontoise, France
[2] LDFC, Université Louis Pasteur, 67084 Strasbourg, France
[3] Dep. of Applied Mathematics, RSPHYSSE, Australian National University, ACT 0200 Canberra, Australia

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Abstract. A foam is a space-filling cellular pattern, that can be decomposed into successive layers or strata. Each layer contains all cells at the same topological distance to an origin (cell, cluster of cells, or basal layer). The disorder of the underlying structure imposes a characteristic roughening of the layers. In this paper, stratifications are described as the results a deterministic "invasion" process started from different origins in the same, given foam. We compare different stratifications of the same foam. Our main results are 1) hysteresis and 2) convergence in the sequence of layers. 1) If the progression direction is reversed, the layers in the up and down sequences differ (irreversibility of the invasion process); nevertheless, going back up, the layers return exactly to the top profile. This hysteresis phenomenon is established rigorously from elementary properties of graphs and processes. 2) Layer sequences based on different origins (*e.g.* different starting cells) converge, in cylindrical geometry. Jogs in layers may be represented as pairs of opposite dislocations, that move erratically because the underlying structure is disordered, and end up annihilating when colliding. Convergence is demonstrated and quantified by numerical simulations on a two dimensional columnar model.

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1 Introduction

Foams and other disordered cellular structures have strong structural similarities [1,2], although they are made of different materials (liquid froths, metallic or polymeric foams, etc.). Even if universality is far from firmly established for all foams, many features manifestly depend neither on the constituting materials, on the forces between them, nor on the length scales involved. It is therefore natural to describe foams and random cellular patterns at the level of topology.

To account for correlations and statistics beyond the one-body properties (concentrations), we must introduce a topological distance. In foams, nearest neighbour cells are clearly defined as sharing an interface. In the dual network, the cells are represented by points connected by links, one for each facet in real space. Thus the dual of a foam is a graph, in which a topological distance is defined as the number of links (edges) in the shortest path connecting two points (vertices) [3–5,7].

Covalent structures, like those occurring in glassy materials, are also described in terms of graphs, but in real space: The atoms, molecules or clusters sit at vertices and the covalent bonds define the edges. Since covalent interactions are carried by quantum electron clouds, defining the bonds may not be always free from ambiguities. This is even more so when non covalent interactions are involved. In these cases, unambiguous geometric constructions, such as the Dirichlet-Voronoi one, complement or replace physico-chemical criteria. But all these structures, even covalent glasses [8], are foams, space-filling cellular patterns.

A layer is the set of vertices / cells at a fixed topological distance j from an origin O. A partition of the whole foam into successive layers $j = 1, 2, 3, \dots$ is a stratification of the cellular pattern.

There are many reasons for improving our understanding of stratifications. The number of nodes / cells, in successive layers —the population, for short— gives almost the same information as the pair correlation function [4,7,9]. As is well known, the correlations are related to the response of the system to all kinds of solicitations. In disordered materials, this question is of particular interest: is the response coded in the geometry and how? Conversely, beyond elasticity, external actions may modify the structure. How? Aging is a common characteristic

[a] e-mail: oguey@ptm.u-cergy.fr
[b] CNRS & UCP UMR 8089

of glassy materials which almost never reach equilibrium; aging may occur spontaneously or under an external influence, and often inhomogeneously.

All these questions involve structure. Our purpose, here, is to analyse some of the fundamental geometric tools, to set the ground for finer investigations. Ultimately, an energy should be introduced. But, in complex systems, the step from geometry to energy is often easier than understanding the geometry. Foams are paradigmatic in this respect: in a first approximation, energy is edge length (in 2D) or interface area (in 3D) times a constant (the surface tension).

Viewed as dynamical processes, considering j as time, the layer sequences $j = 1, 2, ...$ represent the successive stages of signals, fronts, epidemics, propagating at unit velocity. There is a close analogy with aggregation-deposition and related problems (*cf.* [9] and Refs. therein).

One of the differences, however, is that the underlying foam is given *a priori* in its full integrity. The stratification is an additional structure —an ordered partition— of the foam. It is therefore necessary to disentangle what is general to layers, from what depends specifically on the underlying foam. Notably, the same foam structure can have many possible stratifications. Convergence will enable us to classify these stratifications.

The arbitrariness comes from the choice of the origin. Which symmetry, in a given foam, encompasses these equivalent choices? For example, in [4], the leading asymptotic behaviour of the layer population K_j was found (numerically) to be independent of the central cell (the origin). Here, we give a strong support to this numerical result by showing that the stratifications do actually converge. The central cell may even be replaced, as the origin, by a cluster of cells.

Another difference between stratifications and aggregation-deposition, is of importance: The elementary models of aggregation, such as the Eden model, are random processes on regular lattices [10]. In our case, the underlying structure —a foam— is random, with quenched disorder, whereas the process —stratifying— is deterministic, without any randomness. Some geometrical features, such as roughness, are similar in both types of systems [9]. In the present paper, we insist on the aspects which are specific to the second class.

Irreversibility and hysteresis are presented in Section 2 and analysed in Section 3. Convergence is introduced in Section 4; followed by quantitative results, which are related to roughening in growth models. Section 5 is a discussion of our results. The precise definitions are given in Sections 1.1 and 3.

1.1 Layers

The *topological distance*[1] between two vertices in a graph is the minimal number of edges needed to connect them. It will be called simply distance hereafter, since no other notion of distance will be considered.

[1] Also called graph distance in mathematical literature.

The dual of a cellular structure is a graph. Neighbouring cells, sharing an interface, are at distance 1 from each other. In almost all natural foams, two neighbour cells have only one interface in common, so that the dual graph is a simple graph. (In general, the dual of a cellular structure can be a multi-graph, but this does not affect the distance between two cells.)

In a foam, the cells are classified in layers, through topological distance: Given a (connected) set O of cells, *layer* number j (layer j or lay(j) for short) is the set of cells at a distance j from O.

The origin, $O = \text{lay}(0)$, is a set of cells which uniquely defines the stratification. O can be a single cell, a (simply connected) cluster of cells (concentric geometry) or a connected set such as a row of cells (infinite in open, Euclidean geometry, or going once around the cylinder in cylindrical geometry).

Layers can be defined directly in the cellular network, as follows [6,7,12]. The origin O constitutes layer $j = 0$. Cells not in O but in contact with O constitute the first layer. Then, inductively for $j = 1, 2, \ldots$, layer j is made of all the cells, not yet counted, in contact with layer $j - 1$.

If, as in modelling chemical structures, the origin is a single vertex —an atom in the compound—, then the layers are coordination shells [13–15]. So layers, coronas [16] and coordination shells are synonyms. We also name them strata, because they partition the foam —the set of vertices in the dual— into an ordered collection of subsets, making altogether a *stratification* (or foliation).

Most often, we think of the embedding space as either Euclidean (the plane in 2D) or a (semi-infinite) cylinder equivalent to a domain of bounded base with periodic boundary conditions in the x direction(s) (a circle in 2D) and infinite along the axis of the cylinder (coordinate $y > 0$).

Remark In principle, since stratifications are defined in terms of intrinsic notions such as subsets and distance, embedding plays no role for the matters treated in Sections 2 and 3 (hysteresis). Even if our illustrations and simulations are in 2D, most of our results hold for foams in any dimension.

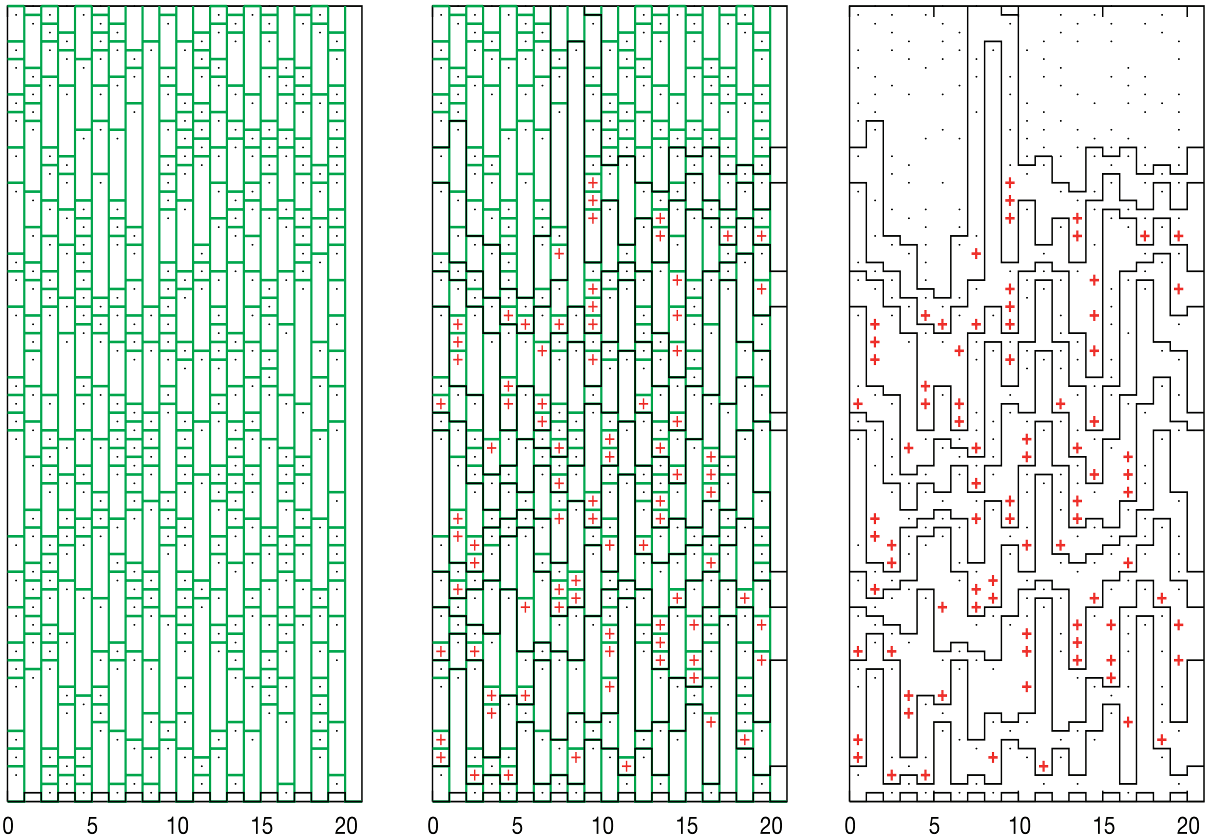
But a foam is, physically, a space-filling cellular pattern, which is, and must be represented as, embedded.

In layer j,

- all the cells are neighbours to at least one cell in lay($j-1$);
- many, but not all cells, called *regular*, are also neighbours of cells in lay($j+1$);
- the cells which do not share an interface with any cell in lay($j+1$), are called *defects* or *inclusions*.

The first statement is a condition for membership of layer j. The others are definitions of regular cells and defects in layer j. We shall see in another paper that defects are sources of frustration, curvature and non-triviality of the stratification.

In summary, a *stratification* $\ell = \{\ell_j\}_{j \geq 0}$ is a partition of the foam (of the vertices in graphs) into layers — the



Bare cells of the columnar foam; top square of each cell marked by a dot. First layers superposed on the cells; the defects are now marked by '+'. Layer boundaries only; the regular cells remain dots '·'.

Fig. 1. Graphical conventions for displaying cells, defects and layers, shown on an example with $L = 20$.

strata ℓ_j, $j = 0, 1, 2, \ldots$. Each layer is the set of cells c (vertices) at distance j from O: $\ell_j = \{c \mid \mathrm{dist}(c, O) = j\}$.

Shell h_j is defined as the outer boundary of layer j; it is the bounding contour of interfaces separating cells in layer j from cells in layer $j + 1$.

1.2 The columnar model

This toy model (the columns) is a useful laboratory for the structure of foams and as a model of growth. It is a lattice version of the Poisson partition of Fortes [17]. We use it for illustration and to get quantitative estimates on roughness and convergence where analytical results are still out of reach (Sect. 4). Otherwise, most of the features presented here are valid generally, not limited to this example.

The relevance and limits of this model were discussed in [9]. We recall the definitions here.

The model is a 2D packing of columnar cells, each of width 1 (in the horizontal, x, direction) and of random length s (height in the vertical, y, direction). The sizes (s is both length and area) of the individual cells are taken as independent random even numbers, identically distributed with exponential law:

$$\Pr(s) = \frac{1-q}{q}\, q^{s/2},\ s = 2, 4, 6 \ldots. \qquad (1)$$

The parameter q has a fixed value in $]0, 1[$. It controls the mean cell size through $\langle s \rangle = 2/(1 - q)$. Throughout the paper, we will take $q = 1/2$, $\langle s \rangle = 4$.

The foam lies on a semi-infinite vertical cylinder, meaning that it is periodic, with period L, in the x direction. The height s of each cell is an *even* random number. With ground $y_0(x) = x \bmod 2$ (crenellated profile), this ensures that the vertices have coordination 3, as in real foams. The system is unbounded in the positive y direction.

To avoid overloading the pictures, the graphical convention of Figure 1 (right) will be used: the cell boundaries are not drawn; the lines are layer boundaries (= shells $h_j, j = 0, 1, 2, 3, \ldots$). The top square of each cell is marked by a dot (.) when the cell is regular, by a cross (+) when it is a defect [3,7,9].

2 Up and down: irreversibility

We compare different stratifications on the *same* cellular pattern. In this section, we build two sets of layers, one with distance increasing upwards (stratification from the bottom up), the other with distance increasing downwards (stratification from an origin at the top). Later, in Section 4, we compare stratifications rising in the same

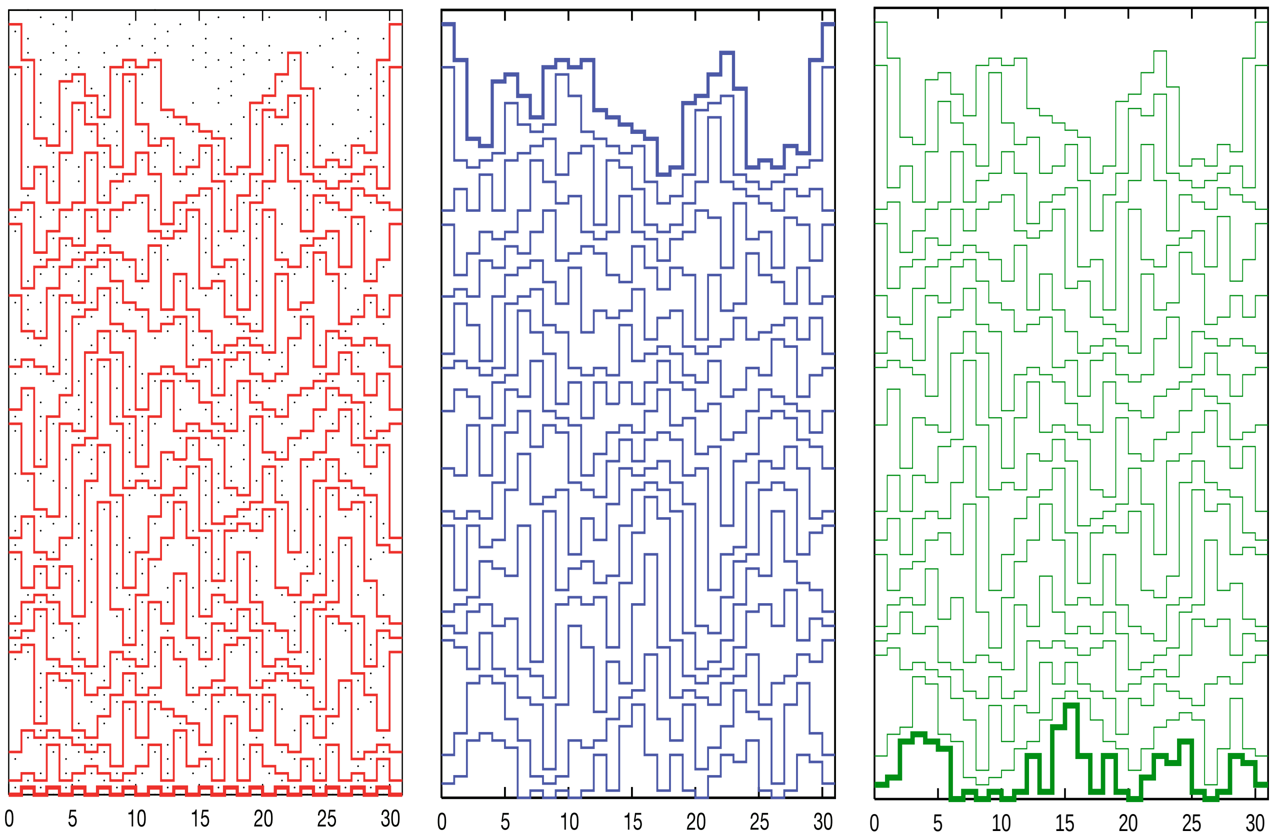


Fig. 2. $L = 30$: stratifications of 15 layers up, down and up again. The ground $j = 0$ is highlighted in each case. The profiles $h_j(x)$ of the two upwards stratifications are identical for $j \geq 12$.

direction but based on different origins (grounds). The question is whether they match, and, if so, how?

2.1 Up and down

Starting from an origin $A_0 = O$ or a ground h_0, if we build the layers upwards $A = \{A_1, A_2, \ldots, A_j, \ldots\}$, stop at some $j = d$ and then, taking layer A_d as a new origin O' (equivalent to setting shell h_{d-1} as a starting profile h'_0), build new layers $A' = \{A'_0 = O', A'_1, A'_2, \ldots\}$ downwards, this new stratification A' does *not* coincide with the former (even if we compare just the regular parts). The top most layer of A coincides with the origin of A', by construction. But then, some cells, qualified as defects in upward layers, switch to being regular in downward layers and *vice versa*, etc. Since these switches cumulate during buildup of the stratification, we may expect that the coherence between the two reverse stratifications A and A' is rapidly lost. This appears to be the case, at first sight (see Fig. 2 left and middle).

Notably, the last shell going down, h'_{d-1}, is different from the upward starting ground h_0. In simple cases as the one illustrated here, it lies in a neighbourhood of h_0, but it is different. This difference will be described in Section 3.

2.2 Back up

What happens if we go back again? Take the last down shell h'_{d-1} as a new ground, h_0^B, and build another stratification $B = \{B_0 = A'_d, B_1, B_2, \ldots\}$ climbing up again. This third stratification $B = \{B_j\}$ is different from the first two: different from A' because of irreversibility; different from A because the new ground, h_0^B, is not, in general, a shell h_j of the first stratification (Fig. 2 right).

Nevertheless, after climbing up, building B up to B_d, the last profile fits exactly the same profile as the first crest: $h_{d-1}^B = h_{d-1}$, $B_d = A_d$. This will be proved later.

Further up and down processes repeat A' and B. Indeed, the next stratification B' (downwards) is degenerate with A' since it starts from the same origin, and so on. Recall that all these stratifications are based on the same, fixed, but random, foam.

2.3 Hysteresis

We have therefore a *hysteresis* cycle, caused by the presence of defects. Indeed, defect-free stratifications are reversible. Examples of these are the rows and columns parallel to the square basis in le Caer's construction [18,19], the vertical columns in the columnar model, or even the horizontal layers $\{\ell_j^0\}$ after defect coalescence.

All these are flat or pure gauge models like the Mattis model for spin glasses [20]. But, let us stress this point, these models admitting defect-free stratifications are not generic. Notably, in le Caer's model, there are special correlations between neighbouring cells [19]. Topologically random foams are not flat. Hysteresis might even be taken as a measure of non trivial disorder.

3 The geometry of layers

In this section, we present the stratifications as analogous to foliated structures. In particular, a proof is given of the fact that the extreme layers are exactly recovered by the down-up procedure.

3.1 Layers as sets

The *distance between sets* A, B is defined as

$$\mathrm{dist}(A,B) = \min\{\mathrm{dist}(a,b) | a \in A, b \in B\}. \qquad (2)$$

In this sense, layer j, as a set of cells, is at distance j from the origin O which can consist of more than a single cell. In fact, by definition, *all* the cells of $\mathrm{lay}(j)$ are at distance j from O. But the converse is not true: *not* all the cells of O are at minimal distance j from $\mathrm{lay}(j)$. This is a first indication of irreversibility.

3.2 Geodesic sections

Let us go on. By definition, any cell c_j of $\mathrm{lay}(j)$ has at least one neighbour c_{j-1} in $\mathrm{lay}(j-1)$; c_{j-1} has a neighbour c_{j-2} in $\mathrm{lay}(j-2)$ etc. down to some *root* cell c_0 in O. Thus, there is always at least one *connected section* linking any cell c_j in layer j to O. In the dual, these connected sections are lines of minimal length $j = \mathrm{dist}(c_j, O)$, *i.e.* topological *geodesics*, linking O and $\mathrm{lay}(j)$. Moreover, these sections consist of regular cells exclusively. In particular, defects in O cannot be root cells.

Stratification is analogous to foliation in differential geometry. The layers are the leaves, and any section can serve as base space (isomorphic to $\mathbb{Z}$ or some subinterval of $\mathbb{Z}$).

The layer structure is robust along these sections: there is exactly one cell per layer crossed. Along these lines, each step is a move from a layer to the next one — upward or downward. Thus, the sequence of layer numbers $j = 1, 2, \ldots$ coincides with topological distance along these lines (counted, respectively, from bottom up, or from top down). The set of linking geodesics constitutes an orthogonal *skeleton* for the stratification.

As already stated, there is a section linked to every cell in $\mathrm{lay}(j)$, but not every cell o of O is at distance j from $\mathrm{lay}(j)$; only the root cells are. Linking geodesics starting from different top cells may fuse on the way down. So the whole set (of linking geodesics) is a forest with branches attached to every cell of $\mathrm{lay}(j)$ but only a few root cells in $\mathrm{lay}(0) = O$. (Note that there may be more than one geodesic connecting two given cells.) Another forest pattern was introduced in [7].

The space left —that is, the part of the foam not covered by the skeleton— is the place where irreversibility occurs; the layers down differ from the layers up.

3.3 Up and down revisited: parallel layers

Two sets A and B are *parallel* if there is a positive number d such that i) all the $a \in A$ are at the same distance d to B and ii) all $b \in B$ are at the same distance d to A.

With respect to stratifications, where the sets are sets of cells and distance is topological distance, parallel sets enjoy special properties. If A and B are parallel at distance d:

- In the stratification based on A, B is a subset of the dth layer: $\mathrm{lay}(0) = A \Rightarrow B \subset \mathrm{lay}(d)$. Moreover, all the cells of A are root cells.
- Conversely, A is a subset of the dth layer of the stratification based on B: $\mathrm{lay}(0) = B \Rightarrow A \subset \mathrm{lay}(d)$. All the cells of B are root cells in this stratification.

The inclusions are weak. Indeed, we will shortly see cases where the "$\subset$" reduces to an equality. Nevertheless, in general, $\mathrm{lay}(d)$ based on A —in the first point— may contain other cells than those of B: for example, two sets $A = \{a\}$ and $B = \{b\}$, containing a single cell each, are trivially parallel but, with $d = \mathrm{dist}(a, b) > 0$, the d'th layer around either $\{a\}$ or $\{b\}$ contains other cells than b or a respectively (provided the foam does not reduce to a single row).

In the notations of Section 2, we now show that the layers A_d and A'_d are parallel at distance d. The proof requires a few basic (in)equalities.

Up ↑

In the up stratification $A_0 = O, .., A_j, ..$ —as in any stratification— $\mathrm{dist}(\mathrm{lay}(j), O) = j$ only implies $\mathrm{dist}(\mathrm{lay}(j), o) \geq j$ for an arbitrary cell o of O. Equality holds if and only if o is a root cell c_0 for some geodesic section. Moreover, equality must hold for at least one cell; there always is at least one root cell in O.

Down ↓

Consider the down stratification $A'_0, A'_1, A'_2, \ldots, A'_d$. A'_0 consists of all the cells of A_d of the up stratification. Call $\{b_i\}$ the cells of layer A'_d. $\{b_i\}$ includes all the root cells of O. All the others must lye below $\{b_i\}$ because they satisfy strict inequality: $\mathrm{dist}(o, A_d) > d$.

Parallelism ↕

Thus, A'_0 and A'_d are parallel.

Indeed, i) $\mathrm{dist}(b, A'_0) = d,\ \forall\, b \in A'_d$ holds by definition of layer A'_d. To see that ii) $\mathrm{dist}(a, A'_d) = d,\ \forall\, a \in A'_0$, note that the construction of the layers implies $\mathrm{dist}(a, A'_d) \geq d$. On the other hand, in the up stratification, there is a cell $c_0 \in O$ at $\mathrm{dist}(a, c_0) = d$; being root, c_0 also belongs to A'_d. Therefore $\mathrm{dist}(a, A'_d) \leq d$, which proves equality ii).

In the middle and right parts of Figure 2, the top and bottom displayed layers are parallel.

Remarks

1. Note that A'_0, which was set equal to A_d, contains no defect for the downward stratification. Indeed, any cell $c \in A_d$ is at distance 1 of a (regular) cell of A_{d-1} and any regular cell of A_{d-1} is reached this way, implying $A^{\mathrm{reg}}_{d-1} \subset A'_1$. Now a defect c_d in A'_0 would be at distance at least 2 from A'_1, in contradiction with $c_d \in A_d = \{c \,|\, \mathrm{dist}(c, A^{\mathrm{reg}}_{d-1}) = 1\}$.
2. Nothing special is assumed on either the foam or the original layer. The point where we turn back ($j = d$) is chosen arbitrarily; layer A_d is normal, with neither more nor less defects than any other.
3. Strictly following our definitions, in the "down" or backward stratification based on $A'_0 = A_d$ (a single row or layer of the up stratification), the subsequent layers A'_j consist of two parts: one going down as we have described, and another one going further up, equal to A_{d+j}, which we have ignored. Most often, this part goes away and does not interfere with the portion of the foam under consideration. We always assume that the situation is safe in these respects. For foams embedded in spaces of more complicated topology (with handles etc.), not separated by closed contours, additional care should be taken here.
4. Going up and down establishes a 'reciprocity' relation between the layers $A'_0 = A_d$ and A'_d, slightly stronger than parallelism. Such a reciprocity does not hold in general; most often, two layers in the same stratification are not even parallel. In order to get a pair of reciprocal layers, a precise procedure must be followed, such as the up-down trick.
5. A stack of d successive layers has minimal thickness when it is delimited by a pair of reciprocal layers. With the previous notations, this means $\mathrm{dist}(o, A_d) \geq d = \mathrm{dist}(c, A_d)$ for all o in A_0, c in A'_d, whatever A_0 we start from.
6. Reciprocity does not mean reversibility of the process (layer sequence). The situation is typical of hysteresis. It is more easily described by viewing stratifications as paths in the set of sets of cells. Two layers in 'reciprocity situation' are connected by two paths, one up and one down, which follow different trajectories despite the fact that they have the same endpoints —the two reciprocal layers. Starting from one end layer (0 or d) will give the other exactly, in d steps. Nevertheless, the collections of layers, between 0 and d, are in general different in the up and down stratifications; most of the intermediate layers j differ from their partners $d - j'$ as sets of cells. The difference is not just a trivial numbering reversal.
7. Reciprocity does not coincide with parallelism. It implies parallelism, but parallelism is weaker because it misses some completeness condition. Indeed, if A and B are parallel sets, B is contained in some layer, say lay(d), of the stratification based on lay(0)=A. But, in general, B is only a subset of lay(d), whereas our condition of reciprocity involves complete layers. Similarly, A may only be a subset of the d'th layer from B or lay(d), as shown in the following example.

An example: concentric stratifications

Take a single cell $O = \{o\}$ as origin and build the concentric stratification around it. Then o must be a root cell. Going up and down (out and in, implying a new stratification inwards) brings one back to a cluster C containing the starting cell o possibly surrounded by other cells. Lay(j), which, in this case, is the topological circle of radius j, and its centre $\{o\}$ are parallel (according to our definition). But they are not 'reciprocal'; up-down does not come back to only $\{o\}$. Cluster C, on the other hand, is both parallel to, and in reciprocity relation with, the topological circle since it was constructed so.

Notice that o is parallel to any topological circle around it (any azimuthal layer at distance $j = 1, 2, \ldots$). But cluster C is parallel only to some specific circle(s), where the turn back is done, or could be done, in order to get C exactly.

Irreversibility, or hysteresis, is the fact that, in between C and the circle j, the outwards and inwards stratifications are different; this is visible only if $j > 2$.

4 Convergence of the stratifications, dependence on ground

The choice of the origin O is arbitrary. One may choose a single cell, and obtain concentric layers. But choosing a horizontal ground is better adapted to cylindrical geometry. Consider a definite foam on a cylinder. Call $A = \{A_j\}_{j\geq 0}$ the stratification based on $y = h^A_0(x) \simeq 0$ (a connected set of cell boundaries; y is the coordinate along the cylinder axis).

For the same foam, we could take as origin another profile $\{h^B_0(x) \mid x = 0, .., L-1\}$ following other cell edges. Let $B = \{B_j\}_{j\geq 0}$ be the stratification based on h^B_0. How do A and B compare ?

Global shift

If h^B_0 is a shell of A, say $h^B_0 = h^A_k$ for some integer k, then, trivially, $B_j = A_{j+k},\ \forall\, j > 0$. The two layer sequences are identical; only their label differ by an integer k (an irrelevant phase shift). Therefore, only profiles h_0 with centre of mass near $y = 0$ need be considered.

Convergence

From numerical simulations on columns and topological foams with randomly generated h_0^B, we observe that the stratifications $\{A_j\}, \{B_j\}$ converge: for any h_0^B, there are integers J, k such that $B_j = A_{j+k}$ for all $j > J$.

The rate of convergence will be discussed later (Sect. 4.3). First, we analyse the phenomenon in terms of dislocations.

4.1 Dislocation pairs in the stratifications

Apart from the flat ground $h_0^A(x) = 0$, the simplest starting ground is a 'podium': $h_0^B(x) = 1$ for $x_+ < x < x_-$, $= 0$ otherwise (in vertical units of layers). The steps at x_+, x_- are a pair of dislocations in the stratification, with strengths $+1$, -1. Because of periodic boundary conditions, the strengths must sum up to 0.

Let us compare A, based on a fixed ground, with a stratifications B, based on a podium h_0^B of width w and of height 1 in units of A-layer thickness. At $j = 0$ the dislocations are at the ends of the podium: $x_+(0)$ and $x_-(0)$ with $|x_+(0) - x_-(0)| = w$. Choosing the maximal distance $w \simeq L/2$ will give an estimate of the convergence time for more general situations.

At any later 'time' j, away from the dislocations $x_\pm(j)$, the two layer systems (profiles, inclusions, etc.) are the same, except for a shift of 1 in numbering between the two dislocations. The differences are confined to the region near x_+ and x_- where the numbering makes steps.

Figure 3 is based on the fact that, for many cells, the regular/defect qualification strongly depends on the stratification. Therefore defect cells can serve as "tracers" to compare two stratifications of the same foam: in Figure 3, the marks indicate cells which are defects in one stratification but not in the other. These differences look like two random walks which ultimately annihilate, as in a 'diffusion-reaction' phenomenon.

The convergence occurs at time of first collision J, when the opposite dislocations meet for the first time and cancel. The layers agree from there on because, for a fixed underlying foam, the process

$$\ldots \rightarrow \mathrm{lay}(j-1) \rightarrow \mathrm{lay}(j) \rightarrow \mathrm{lay}(j+1) \rightarrow \ldots$$

is deterministic.

Due to periodicity in the x direction, the dislocations may fuse on one side (with vanishing $h_j^B - h_j^A > 0$ region), or on the other (vanishing $h_j^B - h_j^A = 0$ region). Convergence means that $A_j = B_j$ in the former case, $B_j = A_{j+1}$ in the latter, for $j > J$.

Incidentally, in a crystalline foam, the analogous trajectories would be periodic in space (ballistic regime). Therefore convergence would occur in time $j = J$ linear in L (or not at all, when the lines x_+, x_- are parallel).

Remark The various stratifications are made over a given, random structure. Drawing the successive layers is therefore an entirely *deterministic* process over the same random structure. Convergence is like many of these mechanisms for finding successive key cards in a given, shuffled pack. Once two stratifications are in phase at some time J, they remain in phase thereafter.

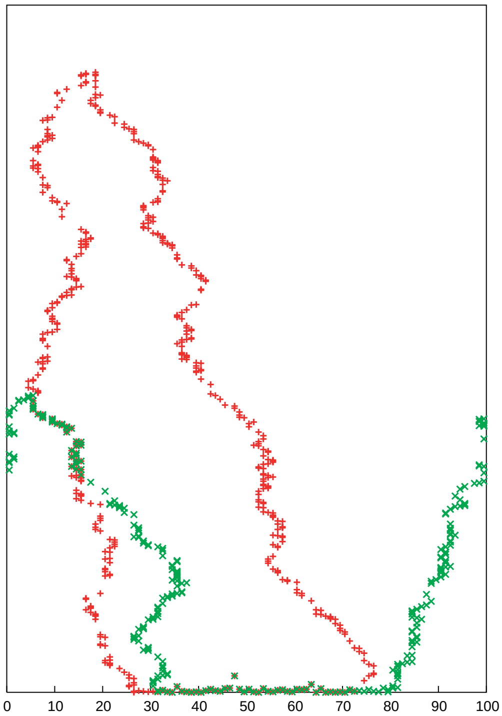


Fig. 3. Two examples of layer convergence. For a given, columnar foam, we compare two stratifications: B is based on a podium at $j = 0$; the reference A is based on a flat ground. Only the cells which are defect in one stratification (A or B) but not in the other are marked. The podium has width $w = L/2$, and centre at $L/2$ in one case (+) and at $L/2 + 6$ in the other (×). In both cases, the pair of dislocations annihilates at some time $j = J$ (different in each case). In the first case, the layers end up in phase. In the second case, the final time shift is one (as if the podium had covered a full layer). The sample contains $(L = 100) \times 400$ cells.

Because of periodic boundary conditions, a general ground h_0 can always be decomposed into dislocation pairs ($+/-1$ steps). When, initially, there is a large density of dislocations (highly corrugated h_0^B)), many dislocation pairs cancel at small j because the partners are initially close to each other; this holds for random (diffusion) and crystalline (ballistic) foams. The ultimate convergence of the stratifications is controlled by the few dislocations that survive at longer time (j). This is further analysed in Section 4.3.

4.2 Attractor

Clearly, the outcome of the convergence is a layer system —a stratification— which is a stable attractor.

Specifically, there are two stationary stratifications: one up and one down.

4.3 Convergence rate

In random foams, convergence depends on disorder and correlations. Qualitatively, if the random motion of the

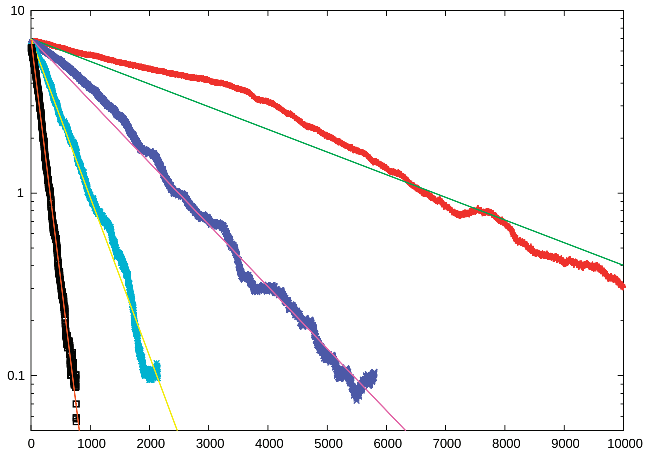


Fig. 4. $\langle \Delta h_j \rangle = \langle \min_k |h_j^B - h_{j+k}^A| \rangle$ as a function of j for $L = 100, 200, 400, 800$. (Average over 50 stratifications $B \times 50$ different foam structures). Fits are of the form $c \exp(-j/\tau)$. For each foam, the 50 stratifications B have randomly chosen grounds. Each stratification contains more than 10 000 layers.

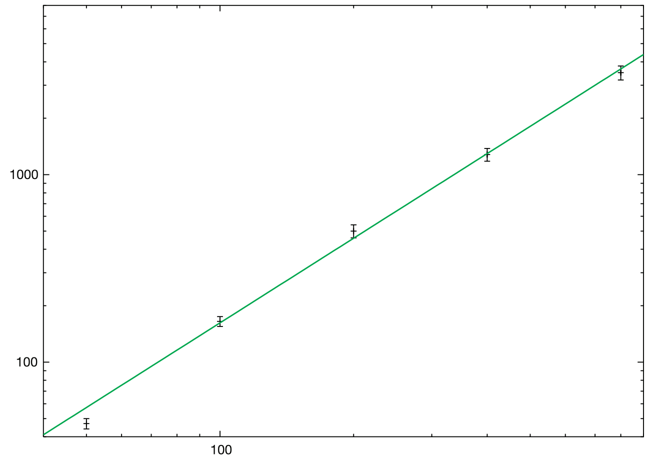


Fig. 5. Correlation time τ in function of L, log-log. The line is $0.162 \times L^{1.5}$.

dislocations is governed by some cooperative phenomena related to roughening, we can guess that the relative diffusion will spread at the same rate as roughness. This implies that the expected convergence time τ should be of the same order as the time needed to reach $\xi_j = L$, where ξ_j is the roughness correlation length along the jth layer (see [9,11] or [22]) and $L/2$ is the initial mean distance between the diffusing dislocations. In other words, τ would be given by $\tau^{1/z} = O(L)$, z being the dynamic exponent. Our numerical simulations fully support this conjecture, as we now show.

First, at a fixed sample width L, convergence occurs at an exponential rate. This has been checked by measuring the mean distance between the profiles, $\langle \Delta h_j \rangle$, where $\Delta h_j = \min_k |h_j^B - h_{j+k}^A|$ (Fig. 4). Indeed, the correlation "time" τ, defined by $\langle \Delta h_j \rangle \propto \exp(-j/\tau)$ as $j \to \infty$, is finite as long as the maximal possible distance between dislocations is bounded, as it is for finite L.

The long time pseudo-diffusion process is manifest in the dependence $\tau(L)$ on sample size L. In the columnar model, which has been shown to fall into the KPZ universality class [9,21], the characteristic "time" $\tau \simeq \langle J \rangle$ scales as $\tau \sim L^z = L^{1.5}$; $z = 1.5$ is the value of the dynamic exponent predicted by KPZ in 2D. This prediction is well confirmed by our simulations. In the range of large L, the plot (Fig. 5) shows a scaling behaviour fitting a power law $\tau \sim L^{1.5}$.

5 Discussion, conclusions, perspectives...

5.1 Summary

For foams or random covalent structures, we have shown that the layer sequences are irreversible. The stratifications in one direction and the other differ even if the two sequences share a whole layer (this can be done by the up-down trick). The hysteresis between up and down stratifications is due to topological defects, inherently present when the disorder in non trivial. The up-down procedure leads to topologically parallel layers at distance d, enjoying special properties.

- Reciprocity: each one may be reached from the other by building a sequence of d layers.
- Minimal thickness of the enclosed stack: any set of d successive layers ending at one of the parallel layers, but based on another initial condition at $j = 0$, will have a thickness larger than the strip bounded by the parallel layers.

On a given fixed foam, the stratifications based on different origins converge to an attractor, one for each of the two directions (in cylindrical geometry). This pair of attractive stratification appears to be specific of the underlying cellular pattern. The characteristic time for convergence, $\tau \sim L^{1.5}$, agrees with KPZ universality class in two dimensions, as long as the probability distribution decays rapidly for cells with a large number of sides n (exponentially, or as $n^{-\kappa}$ with κ large enough) [23]. This has been confirmed by numerical simulations on the columnar model.

As a consequence, stratifications built on two foam samples differing only by local perturbations —topological transformations like neighbour exchanges, cell birth or coalescence, etc.— will also converge, even if the convergence is, in practical respects, slow (see Sect. 4.3).

As proven in Section 3, the first set of properties — hysteresis, reciprocity, minimal thickness— hold generally, for any type of foam or graph.

Convergence and attractors, however, are still conjectural. They essentially follow from an interplay between determinism of the process and randomness of the landscape. Simulations of rectangular foams with periodic boundary conditions in one direction, infinite in the other direction, confirmed the phenomenon and gave us quantitative results on the rate of convergence, its scaling properties and its relation to roughness.

The main biases of our model are that the disorder is confined to one direction and that the width of the system is finite. These two aspects, local and global, deserve separate discussions.

5.2 Anisotropy

We think that the columnar nature of our model has negligible influence on our observations; our conclusions hold more generally. Convergence was probed in the disordered (vertical) direction, where randomness provides a good imitation of more realistic foams.

Preliminary simulations of topological foams — generated by operating a large number of randomly distributed topological transformations as in [5,24]— show the same properties as those observed in the rectangular model: hysteresis, of course, but also, to some extent, convergence of stratifications, etc.

Notice that convergence is observable and measurable in any type of foam, not only columnar. This is a significant improvement with respect to [9], where most of the analysis was based on height $h(x)$, which is rather specific to the columnar model.

5.3 Boundary conditions

The extension to foams in other types of spaces is twofold.

As already argued, and shown on an example in concentric geometry, parallelism, irreversibility and hysteresis, which can be tested in finite regions, occur quite generally: in planar or 3D foams, embedded in Euclidean or curved spaces.

As to convergence, it holds unambiguously only in cylindrical foams. Cylindrical geometry is quite frequent in condensed matter, zoology, botanic, etc, with numerous examples such as tubules, channels, stems, stalks, straws,... These boundary conditions introduce a definite length-scale, L, into the system. At fixed L, the convergence is exponential in time, or layer number j. The power law with the dynamic exponent describes the asymptote of the characteristic time as a function of L (Fig. 5), in accordance with scaling theory.

When dealing with other boundary conditions, the question of convergence is not straightforward. There are elementary obstructions to the onset of a uniform convergence in concentric geometry. However, convergence may still be true in a weaker sense, either in the mean over each layer, or restricted to sectors of prescribed aperture. All these questions are under current investigations.

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